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An analytical coupled homotopy-variational approach for solving strongly nonlinear differential equation

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ABSTRACT

In the present paper, a novel technique combining the homotopy concept with variational formula has been presented to find accurate analytical solution for nonlinear differential equation with inertia and static non-linearity. The obtained results are compared with other analytical and exact solutions to confirm the excellent accuracy and correctness of the approximate analytical technique. The results of the present paper are valid for large amplitudes of oscillation; also the approximate solutions give excellent result than other methods. We concluded that the first order approximation obtained in current work are almost the same with exact solutions, also works very well for the whole range of initial amplitudes.

1. Introduction

With the fast development of research in nonlinear science, there appears ever increasing interest of scientists and researchers in novel techniques to find efficient methods for obtaining approximate solutions to nonlinear differential equations. The study of given nonlinear oscillator problems is of crucial importance in many branches of sciences such as mechanics, engineering and applied Mathematics. During the recent years, a large variety of new analytical approximate techniques to solve nonlinear differential equations include Energy Balance Method (EBM) [1,2], Amplitude-Frequency Formulation (AFF) [3,4], Variational Iteration Method (VIM) [5], Homotopy Perturbation Method (HPM) [6,7], Homotopy Analysis Method (HAM) [8], Iteration Perturbation Method (IPM) [9,10], Min-Max Approach (MMA) [11], Laplace Transform (LT) [12], Hamiltonian Approach (HA) [13–15], Variational Approach (VA) [16,17], Global Residue Harmonic Balance Method (GRHBM) [18,19], Coupled Homotopy-Variational Approach (CHVA) [20–25] are most accepted approximate methods in studying non-linear models arising in physics, mathematics and engineering that are constantly being developed or applied to more complex non-linear systems.

In recent decades, some researchers have studied the behavior of the current differential equation, for example. Abd El-Latif [26] proposed a new approach by combining the linearization of the governing problem with the (HBM), Molla et al. [27] used the (HBM) to obtain accurate approximate analytical higher-order solutions for nonlinear problem.

In current work, a novel and different technique called the coupled of homotopy with variational approach [5,6] has been employed to find accurate periodic solutions to nonlinear oscillators. The coupled method is applied to derive highly accurate analytical expressions for approximate formulas of frequency.

2. Governing equation of motion

In this section, we consider the vibration of an inextensible clamped-free tapered beam as an interesting and important model for engineering structures with inertia and static nonlinearity in the form [26,27]. Fig. 1 illustrates the physical model of the problem.

$$\frac{d^2x}{dt^2}+x+\alpha x^4\frac{d^2x}{dt^2}+2\alpha\left(\frac{dx}{dt}\right)^2x^3+\beta x^5=0,\ x(0)=A,\ \dot{x}(0)=0. \quad (1)$$

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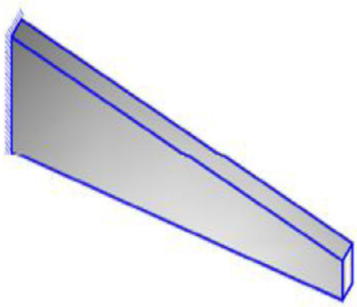

Fig. 1. Geometry of the problem.

Table 1
Comparison of angular frequencies ω_{app} with the exact frequencies ω_{ex} for $\alpha = 1,\ \beta = 1$.

A	ω_2 [26] Error (%)	ω_2 [27] Error (%)	ω Present Error (%)	ω_{ex}
5	2.3701 (1.5821)	2.4397 (1.3080)	2.4613 (2.2050)	2.4082
10	2.3810 (2.5778)	2.4525 (0.3478)	2.4502 (0.2537)	2.4440
15	2.3816 (2.7045)	2.4532 (0.2206)	2.4446 (0.1307)	2.4478
20	2.3817 (2.7401)	2.4533 (0.1838)	2.4495 (0.0286)	2.4488
25	2.3818 (2.7479)	2.4534 (0.1592)	2.4495 (0.0163)	2.4491
30	2.3818 (2.7559)	2.4534 (0.1592)	2.4495 (0.0082)	2.4493
50	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
100	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
200	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
500	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
1000	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
10,000	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495

This system describes the unimodal large-amplitude free vibrsations of a slender inextensible cantilever beam carrying an intermediate mass with a rotary inertia. In Eq. (1), the term $(\alpha x^4(d^2x/dt^2) + 2\alpha(dx/dt)^2x^3)$ represent inertia type fifth nonlinearity from the inextensibility assumption and the term βx^5 is a static-type fifth nonlinearity due to the potential energy stored in bending.

3. Application of coupled homotopy-variational approach

Consider the nonlinear oscillator Eq. (1), the following homotopy can be constructed:

$$x'' + \omega^2 x + p\left[\alpha x^4 x'' + 2\alpha x'^2 x^3 + \beta x^5 + \left(1 - \omega^2\right)x\right] = 0, \quad (2)$$

when $p = 0$, Eq. (2) becomes the linearized differential equation $x'' + \omega^2 x = 0$, and when $p = 1$, Eq. (2) then becomes the original problem. Suppose that the analytical periodic solution to Eq. (2) can be written as a power series in p:

$$x = x_0 + px_1 + p^2x_2 + \ldots. \quad (3)$$

Now, inserting the above Equation into Eq. (2) and collecting terms with the same powers of p, we obtain:

$$p^0: \quad x_0'' + \omega^2 x_0 = 0, \quad (4)$$

Table 2
Comparison of angular frequencies ω_{app} with the exact frequencies ω_{ex} for $\alpha = 1,\ \beta = 2$.

A	ω_2 [26] Error (%)	ω_2 [27] Error (%)	ω Present Error (%)	ω_{ex}
5	2.3701 (1.5821)	2.4397 (1.3080)	2.4613 (2.2050)	2.4082
10	2.3810 (2.5778)	2.4525 (0.3478)	2.4502 (0.2537)	2.4440
15	2.3816 (2.7045)	2.4532 (0.2206)	2.4446 (0.1307)	2.4478
20	2.3817 (2.7401)	2.4533 (0.1838)	2.4495 (0.0286)	2.4488
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500	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
1000	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495
10,000	2.3818 (2.7638)	2.4534 (0.1592)	2.4495 (0.0000)	2.4495

$$p^1: \quad x_1'' + \omega^2 x_1 + \alpha x_0^4 x_0'' + 2\alpha x_0'^2 x_0^3 + \beta x_0^5 + \left(1 - \omega^2\right)x_0 = 0. \quad (5)$$

Solving Eq. (4), we have

$$x_0 = A\ \cos\omega t. \quad (6)$$

The variational approach for x_1 can be obtained as described by [19]:

$$J(x_1) = \int_0^{2\pi/\omega}\left(-\frac{1}{2}x_1'^2 + \frac{1}{2}\omega^2 x_1^2 + \alpha x_0^4 x_0'' x_1 + 2\alpha x_0'^2 x_0^3 x_1 + \beta x_0^5 x_1 + (1 - \omega^2)x_0 x_1\right)dt. \quad (7)$$

In order to improve the solution accuracy, we define a new trail solution in the form:

$$x_1(t) = B\ (\cos(\omega t) - \cos(3\omega t) - \cos(5\omega t)). \quad (8)$$

By inserting Eq. (8) into Eq. (7), we obtain

$$J(A, B, \omega) = \frac{B\pi\left(-64B\omega^2 - 4A\left(\omega^2 - 1\right) + A^5\left(\alpha\omega^2 + \beta\right)\right)}{4\omega} = 0. \quad (9)$$

Setting

$$\frac{\partial J}{\partial B} = 0, \quad \frac{\partial J}{\partial \omega} = 0. \quad (10)$$

Solving the foregoing equations, we successively achieve the value ω as follow:

$$\omega = \sqrt{\frac{3A^4\beta + 12}{\alpha A^4 - 4}}. \quad (11)$$

Therefore, the solution to the first order approximation can be reformed as follows

$$x = A\cos\left(\sqrt{\frac{3A^4\beta + 12}{\alpha A^4 - 4}}t\right). \quad (12)$$

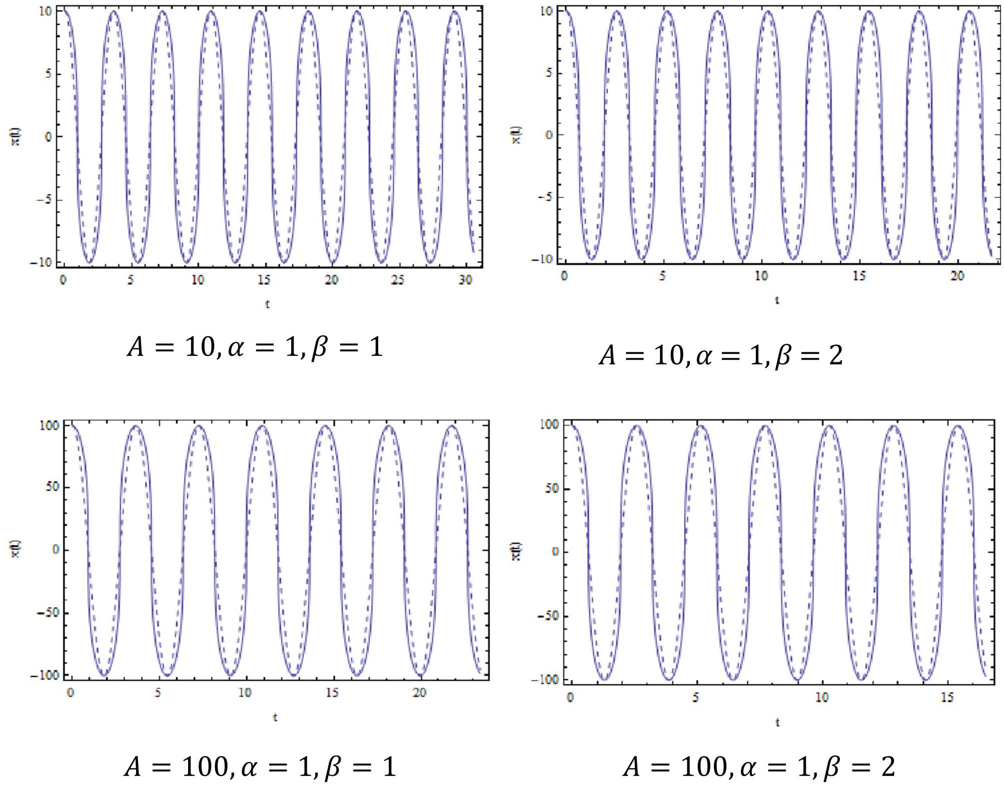

Fig. 2. Comparison of the analytical approximation (- - - -) with the numerical solutions (———).

3.1. Results and discussion of the solution method

The coupled homotopy-variational approach has been applied with a new trial function to obtain the approximated frequency of strongly non-linear equation. We have calculated the first order approximate frequency of Eq. (1) for some particular values of A and compared with exact solution and other analytical solutions obtained by Abd El-Latif [26] and Molla et al. [27], which is presented in Tables 1 and 2 and Fig. 2. The values of the tested example show that the present results are identical with exact solutions. It is worth noting that our findings in comparison with previous works on this subject are highly excellent and involve less complicated computations. It is clear that the first order approximate frequencies are much significantly better result than first and second order approximations [26,27] for large amplitudes of oscillation. As shown in Fig. 2, it is apparent that analytical coupled method has an excellent agreement with the numerical Rung-Kutta method and these expressions are valid for a wide range. As it is clear from Tables 1 and 2, the error of the coupled method is much low than others. Also, it makes the approximate solution rapidly converge.

4. Conclusion

In this estimation, a novel approximate analytical coupled method was employed to establish the relationship between frequency and amplitude of nonlinear equation. The analytical solution obtained in this paper is valid for very large values of the amplitude. It is highly remarkable that a coupled homotopy-variational method provides us with a freedom of choice of trial function. Also, the method is very convenient and in better agreement with the exact and numerical solutions than those predicted by the previously methods, also the method give us excellent result for large amplitudes of nonlinear equation. In addition, analytical solutions presented in the present study give a thoughtful and insightful understanding of the effect of amplitude and parameter.

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References

[1] J.H. He, Preliminary report on the energy balance for nonlinear oscillations, Mech. Res. Commun. 29 (2002) 107–111.
[2] M. Akbarzade, A. Farshidianfar, Nonlinear transversely vibrating beams by the improved energy balance method and the global residue harmonic balance method, Appl. Math. Modell. 45 (2017) 393–404.
[3] Y. Khan, M.K. Yazdi, H. Askari, Z. Saadatnia, Dynamic analysis of generalized conservative nonlinear oscillators via frequency amplitude formulation, Arabian J. Sci. Eng. 38 (2013) 175–179.
[4] A.M. El-Naggar, G.M. Ismail, Applications of He's amplitude-frequency formulation to the free vibration of strongly nonlinear oscillators, Appl. Math. Sci. 6 (2012) 2071–2079.
[5] J.H. He, Variational iteration method-some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3–17.
[6] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non Linear Mech. 35 (2000) 37–43.
[7] V. Marinca, N. Herisanu, Optimal homotopy perturbation method for strongly nonlinear differential equations, Nonlinear Sci. Lett. A 1 (2010) 273–280.
[8] H.M. Sedighi, K.H. Shirazi, J. Zare, An analytic solution of transversal oscillation of quintic non-linear beam with homotopy analysis method, Int. J. Non Linear Mech. 47 (2012) 777–784.
[9] A.M. El-Naggar, G.M. Ismail, Analytical solution of strongly nonlinear Duffing oscillators, Alexandria Eng. J. 55 (2016) 1581–1585.
[10] H.M. Sedighi, K.H. Shirazi, M.A. Attarzadeh, A study on the quintic nonlinear beam vibrations using asymptotic approximate approaches, Acta Astronaut. 91 (2013) 245–250.
[11] H.M. Sedighi, K.H. Shirazi, A. Noghrehabadi, Application of recent powerful analytical approaches on the non-linear vibration of cantilever beams, Int. J. Nonlinear Sci. Numer. Simulat. 13 (2012) 487–494.
[12] S. Kumar, A. Kumar, D. Kumar, J. Singh, A. Singh, Analytical solution of Abel integral equation arising in astrophysics via Laplace transform, J. Egyptian Math. Soc. 23 (2015) 102–107.
[13] M. Bayat, I. Pakar, Nonlinear free vibration analysis of tapered beams by Hamiltonian approach, J. Vibroeng. 13 (2011) 654–661.

[14] Y. Khan, Q. Wu, H. Askari, Z. Saadatnia, M.K. Yazdi, Nonlinear vibration analysis of a rigid rod on a circular surface via Hamiltonian approach, Math. Comput. Appl. 15 (2010) 974–977.
[15] H.M. Sedighi, K.H. Shirazi, Asymptotic approach for nonlinear vibrating beams with saturation type boundary condition, J. Mech. Eng. Sci. 227(2013) 2479–2486.
[16] J.H. He, Variational approach for nonlinear oscillators, Chaos Solit. Fract. 34 (2007) 1430–1439.
[17] Y. Khan, N. Faraz, A. Yildirim, New soliton solutions of the generalized Zakharov equations using He's variational approach, Appl. Math. Lett. 24 (2011) 965–968.
[18] P. Ju, X. Xue, Global residue harmonic balance method for large-amplitude oscillations of a nonlinear system, Appl. Math. Modell. 39 (2015) 449–454.
[19] M. Mohammadian, M. Akbarzade, Higher-order approximate analytical solutions to nonlinear oscillatory systems arising in engineering problems, Archive Appl. Mech. DOI 10.1007/s00419-017-1252-y.
[20] M. Akbarzade, J. Langari, Determination of natural frequencies by coupled method of homotopy perturbation and variational method for strongly nonlinear oscillators, J. Math. Phys. 52 (2011) 023518.
[21] Y. Khan, M. Akbarzade, A. Kargar, Coupling of homotopy and the variational approach for a conservative oscillator with strong odd-nonlinearity, Scientia Iranica A 19 (2012) 417–422.
[22] M. Akbarzade, Y. Khan, Dynamic model of large amplitude non-linear oscillations arising in the structural engineering: analytical solutions, Math. Comput. Modell. 55 (2012) 480–489.
[23] Md.Abdur Razzak, Md.Shamsul Alam, An analytical coupled technique for solving nonlinear large amplitude oscillation of a conservative system with inertia and static non linearity, SpringerPlus 5 (2016) 456–464.
[24] Md.Abdur Razzak, An analytical approximate technique for solving cubic-quintic Duffing oscillator, Alexandria Eng. J. 55 (2016) 2959–2965.
[25] A. Mirzabeigy, M.K. Yazdi, A. Yildirim, Analytical approximations for a conservative nonlinear singular oscillator in plasma physics, J. Egyptian Math. Soc. 20 (2012) 163–166.
[26] G.M. Abd El-Latif, On a problem of large-amplitude oscillation of a non-linear conservative system with inertia and static non-linearity, Appl. Math. Inf. Sci. 1 (2007) 173–183.
[27] M. Helal Uddin Molla, M. Abdur Razzak, M.S. Alam, Harmonic balance method for solving a large-amplitude oscillation of a conservative system with inertia and static non-linearity, Results Phys. 6 (2016) 238–242.